Sec 7.6 Repeated Eigenvalues

Indepence check shortcut:

- Solutions $\underline{X} = e^{\lambda t}\underline{v}$ and $\underline{Y} = e^{\mu t}\underline{w}$ are lin. indep if $\underline{v}$ and $\underline{w}$ are themselves lin indep.

★ · If $\underline{v}$ and $\underline{w}$ are eigenvectors having different eigenvalues, then they are automatically lin indep.

Ex: $\underline{X}' = \begin{bmatrix} 3 & 0 & 0 \\ -4 & 6 & 2 \\ -16 & -15 & -5 \end{bmatrix}\underline{X} \longrightarrow$ know

$\lambda_1 = 0, \quad \underline{v_1} = \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix}$

$\lambda_2 = 1, \quad \underline{v_2} = \begin{bmatrix} 0 \\ -2 \\ 5 \end{bmatrix}$

$\lambda_3 = 3, \quad \underline{v_3} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$

Can make "basis solutions"

$$\underline{X}_1 = e^{\lambda_1 t}\underline{v_1} = \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix}, \quad \underline{X}_2 = e^{t}\begin{bmatrix} 0 \\ -2 \\ 5 \end{bmatrix}, \quad \underline{X}_3 = e^{3t}\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}.$$

$\underline{X}_1, \underline{X}_2, \underline{X}_3$ are automatically lin indep b/c they were made with different eigenvalues. (3x1 system + 3 indep sols ✓)

So gen. solution is $\underline{X} = c_1\underline{X}_1 + c_2\underline{X}_2 + c_3\underline{X}_3$.

Ex ("complete") $\underline{X}' = \begin{bmatrix} 9 & 4 & 0 \\ -6 & -1 & 0 \\ 6 & 4 & 3 \end{bmatrix}\underline{X} \quad (= \underline{A}\underline{X})$

$$\det(\underline{A} - \lambda I) = \cdots = (\lambda-5)(\lambda-3)^2 = 0$$

$\Rightarrow$ eigenvals are $\lambda = 5$ (mult. 1)

$\lambda = 3$ (mult. 2)

Now to find eigenvectors:

<u>Recall/DEF</u> $\underline{\lambda = 5}$: $\mathcal{E}_5 = $ (space of eigenvectors for $\lambda = 5$)

$$\doteq \text{Null}(\underline{A} - 5I)$$

$$(\underline{A} - 5I)\underline{v} = \underline{\emptyset} \longrightarrow \begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \left[\begin{array}{ccc|c} 4 & 4 & 0 & 0 \\ -6 & -6 & 0 & 0 \\ 6 & 4 & -2 & 0 \end{array}\right] \end{array}$$

$$\longrightarrow \cdots \xrightarrow{(\text{RREF})} \left[\begin{array}{ccc|c} \boxed{1} & 0 & -1 & 0 \\ 0 & \boxed{1} & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \leftarrow a = c \\ \leftarrow b = -c \\ c \text{ free} \end{array}$$

so eigenvectors for $\lambda = 5$ are $\begin{bmatrix} a \\ b \\ c \end{bmatrix} = c\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$,

$\mathcal{E}_5 = \left\{ c\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} : c \in \mathbb{R} \right\}$. Pick $c = 1$, $\underline{v_1} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$.

Then $\underline{X}_1 = e^{\lambda_1 t}\underline{v_1} = e^{5t}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ is a solution for $\underline{X}' = \underline{A}\underline{X}$.

$\underline{\lambda = 3}$ $\mathcal{E}_3 = $ (space of eigenvectors for $\lambda = 3$) $= \text{Null}(\underline{A} - 3I)$

$$\longrightarrow \left[\begin{array}{ccc|c} \overset{1}{\cancel{6}} & \overset{2/3}{\cancel{4}} & 0 & 0 \\ \cancel{-6} & \cancel{-4} & 0 & 0 \\ \cancel{6} & 4 & 0 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} \boxed{1} & 2/3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \leftarrow a = \tfrac{-2}{3}b$$

b, c free.

so eigenvectors are $\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -2/3\, b \\ b \\ c \end{bmatrix} = b\begin{bmatrix} -2/3 \\ 1 \\ 0 \end{bmatrix} + c\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$\mathcal{E}_3 = \left\{ b\begin{bmatrix} -2/3 \\ 1 \\ 0 \end{bmatrix} + c\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} : b, c \in \mathbb{R} \right\}$

so let's pick $\underline{v_2} = \begin{bmatrix} -2 \\ 3 \\ 0 \end{bmatrix}$, $\underline{v_3} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$(b = 3, c = 0)$ $\quad$ $(b = 0, c = 1)$

$\underline{v_2}$, $\underline{v_3}$ are lin indep, so solutions

$$\underline{X}_2 = e^{3t}\underline{v_2} = e^{3t}\begin{bmatrix} -2 \\ 3 \\ 0 \end{bmatrix}$$

$$\underline{X}_3 = e^{3t}\underline{v_3} = e^{3t}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

are automatically linearly indep. They are also indep with/from $\underline{X}_1$, because $\underline{X}_1$ was made with a different eigenvalue ($\lambda = 5$).

3 indep solutions $\Longrightarrow$ make general solution

$$\underline{X} = c_1\underline{X}_1 + c_2\underline{X}_2 + c_3\underline{X}_3 .$$

• Notes: char. poly $|\underline{A} - \lambda I|$ was $(\lambda-5)(\lambda-3)^2$

roots were • $\lambda = 5$ with "algebraic" multiplicity 1

(roots were 5, 3, 3) • $\lambda = 3$ " "algebraic" multiplicity 2

DEF: Since $\dim(\mathcal{E}_5) = 1$, we say $\lambda = 5$ has "geometric" multiplicity of 1

[geom. mult. of λ is $\dim(\mathcal{E}_\lambda) = \dim(\text{Null}(\underline{A} - \lambda I))$]

~~DEF~~ ~~Since~~ • So, since $\dim(\mathcal{E}_3) = 2$, $\lambda = 3$ has "geom. mult." equal to 2.

DEF: The "defect" of an eigenvalue λ is

$$d \overset{\text{DEF}}{=} (\text{alg. mult. of } \lambda) - (\text{geom. mult. of } \lambda)$$

In example, both $\lambda = 5$, and $\lambda = 3$ have defect 0.

$d = 1 - 1 = 0$ $d = 2 - 2 = 0$.

We also say "defect 0" $\longleftrightarrow$ "complete"

Ex : (Defect) : $\underline{X}' = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \underline{X}$ (want $\underline{X}_1, \underline{X}_2$ indep)

eigenvalues: $|\underline{A} - \lambda I| = (1-\lambda)^2 = 0$

$\Longrightarrow \lambda = 1$ with alg. mult. 2.

eigenvectors : $\mathcal{E}_1 = \text{Null}(\underline{A} - \underline{I})$

$\longrightarrow$ solve $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\left[\begin{array}{cc|c} 0 & \boxed{1} & 0 \\ 0 & 0 & 0 \end{array}\right] \longleftarrow b = 0$$

free

eigenvectors are $\begin{bmatrix} a \\ b \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$\mathcal{E}_1 = \left\{ a \begin{bmatrix} 1 \\ 0 \end{bmatrix} : a \in \mathbb{R} \right\}$$

so pick $\underline{v_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, make $\underline{X}_1 = e^{1t} \underline{v_1} = e^t \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

But what to do about $\underline{X}_2$?

geom. mult. of $\lambda = 1$ is 1

so <u>defect</u> for $\lambda = 1$ is $d = 2 - 1 = 1$.